The Lee's Amortizer Rule. How to use it?

Miguel R Ramirez. Galena Park High School

This beautiful mechanical analog computer solves problems involving the *Present Value of an Annuity* formula (See Annex A to understand how this formula is obtained):

$$A_n = m \times \frac{1-(1+r)^{-n}}{r} \qquad (1)$$

Where the values represented in this formula are:

A_n = Amount accrued (principal) at the end of n-monthly payments
m = monthly payment
r = Annual interest divided by 12 months
n = number of payments

With electronic calculators with exponential function key x^y, problems involving this formula can be easily solved.

However in the 1940's these gadgets did not exist, and the solution of annuities had to be done using Annuity Tables, but if these were not available, could be solved using a regular slide rule, or in the worst scenario algebraically and with standard logarithmic tables.

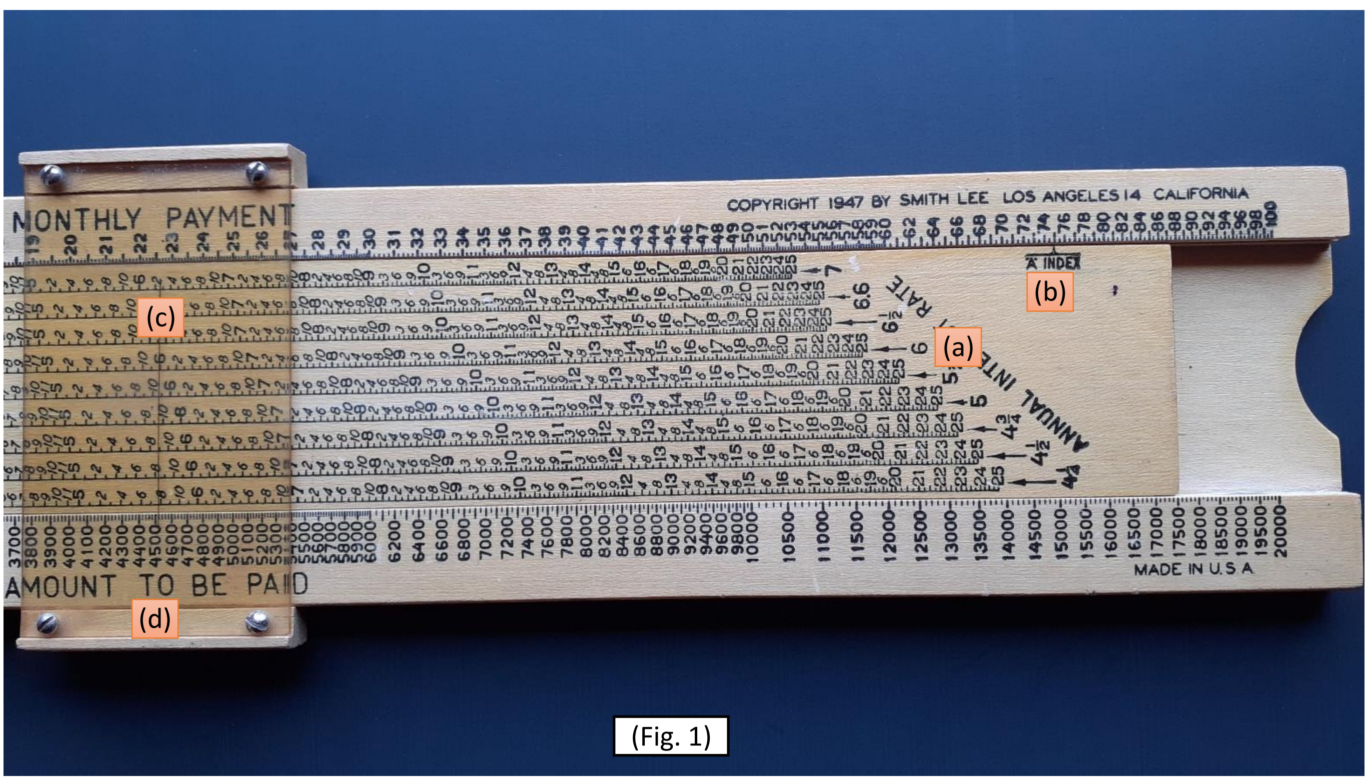
The Lee's Amortizer made the solution of annuities problems very simple. Next examples will illustrate this.

1. What Amount or principal will be amortized with a monthly payment of $75.00 with an annual interest of 6%, after 72 payments?

   Using directly the formula (1) and a calculator:

$$A_n = 75 \times \frac{1-(1+0.06/12)^{-72}}{0.06/12} = 4{,}525.46$$

   The result for this problem is obtained with the Lee's rule after these simple steps, one-setting in slide rule terminology (Fig.1):



(Fig. 1)

(a). The slider has to display the *6%* scale.
(b). Align the A-index with the $75 on the *Monthly Payment* scale.
(c). Move the cursor until hairline is aligned with the 72 (6 years) value of the *6%* scale
(d). Read the accrued value under the hairline on the *Amount to be Paid* scale: $4,525

2. What is the monthly payment if a loan of $7,000.00 is to be amortized in 5 years including an annual interest of 5%

   We have to solve for *m* formula (1):

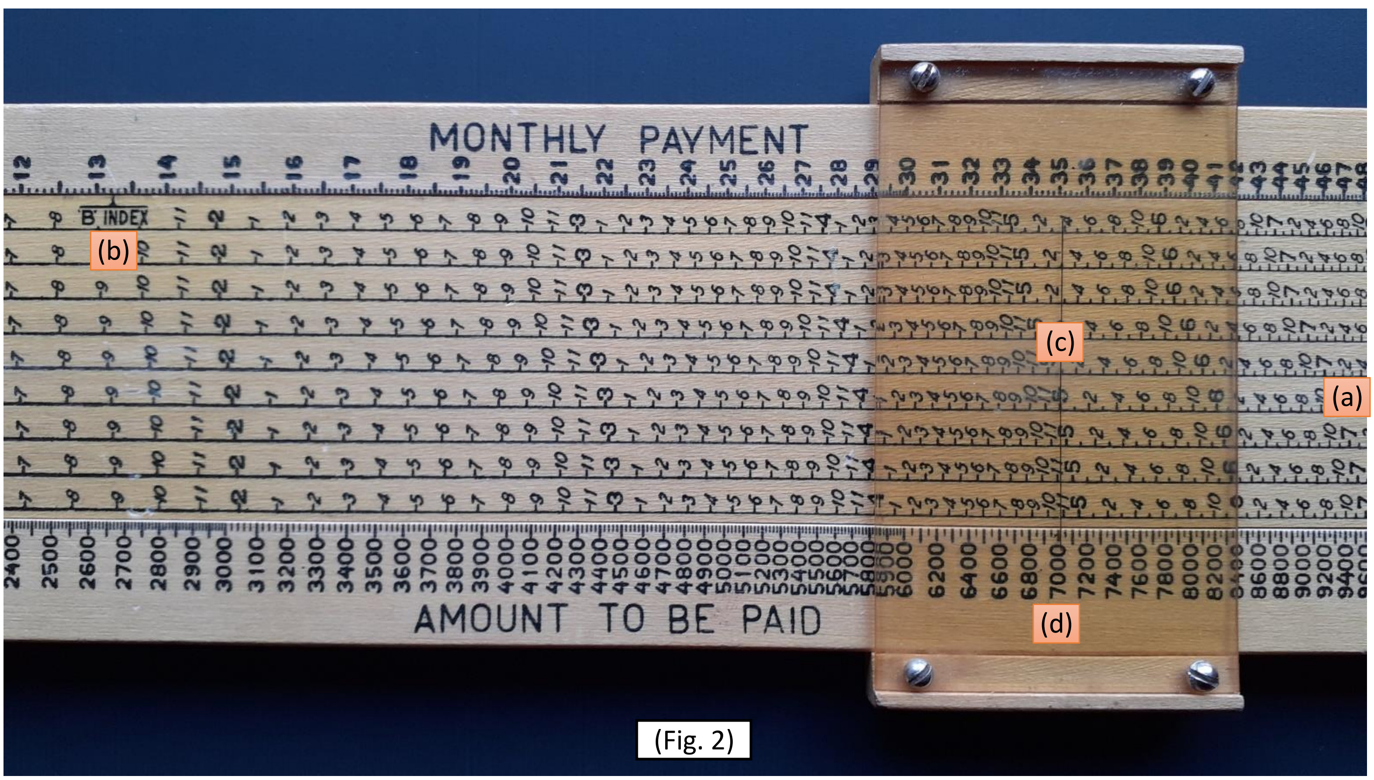
$$m = A_n \times \frac{r}{1-(1+r)^{-n}} \qquad (2)$$

Using directly the formula (2) and a calculator:

$$m = 7000 \times \frac{0.05/12}{1-(1+0.05/12)^{-60}} = \$132.099$$

The result for this problem is obtained with the Lee's rule (Fig. 2) after these simple steps (one setting):

(a). The slider has to display the *5%* scale.
(b). Place the cursor hairline with the $7000 on the *Amount to be Paid* scale.
(c). Move the slider until the value 5 on the 5% is under the hairline
(d). Read the monthly payment pointed by the B-index: $132



(Fig. 2)

3. How many monthly payments are required to amortize a loan of $9,000.00, paying $120.00 each month including an annual interest of 3.5%

   It is necessary now to do some Algebra and use log's to solve equation (1) for *n*:

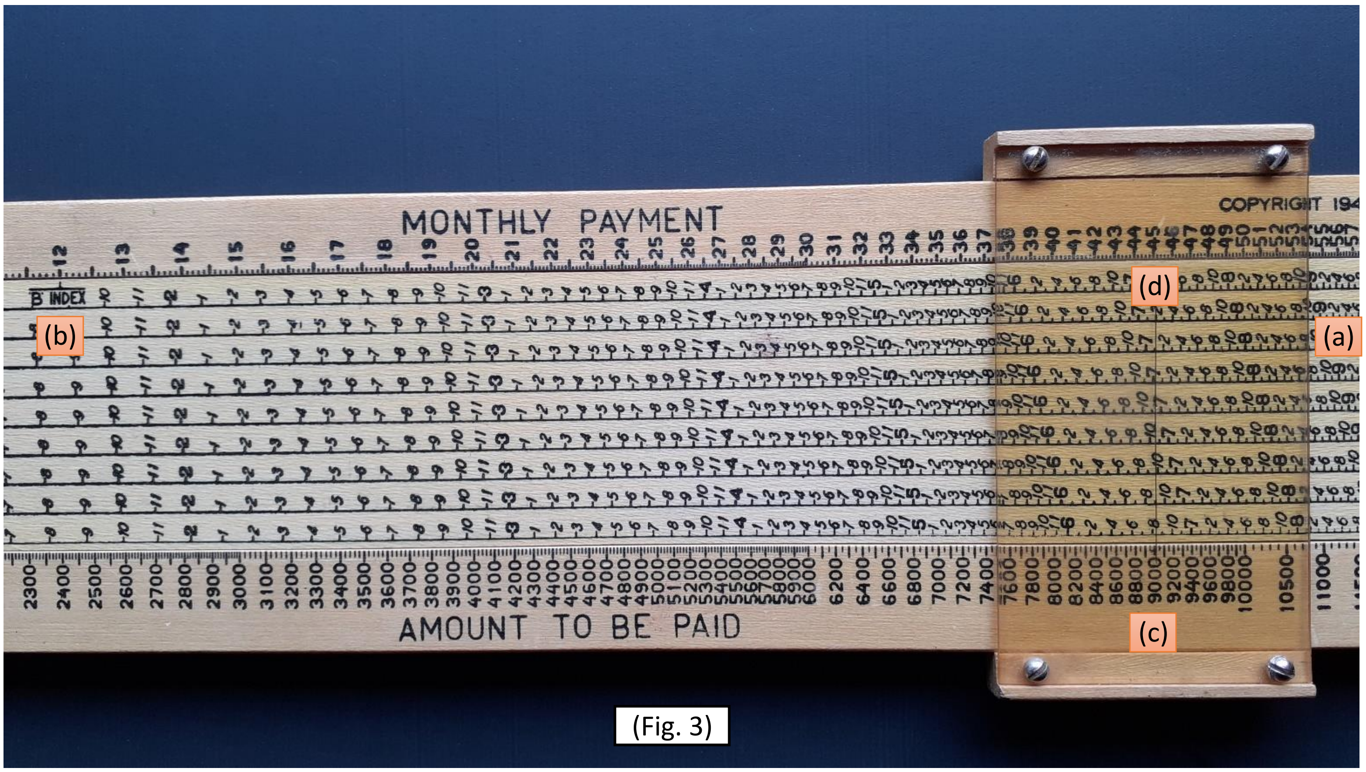
$$n = -\frac{\ln\left(1 - \dfrac{A_n \times r}{m}\right)}{\ln(1+r)} \qquad (3)$$

Using directly the formula (3) and a calculator:

$$n = -\frac{\ln\left(1 - \dfrac{9000 \times 0.035/12}{120}\right)}{\ln(1+.035/12)} = 84.7611\,months$$

The result for this problem is obtained with the Lee's rule (Fig.3) after these simple steps (one setting):

(a). The slider has to display the 3.*5%* scale.
(b). Because $120 is not on the *Monthly Payment* scale, place the B-index under 12, (that will be read 120)
(c). Place the cursor hairline over $9000 on the *Amount to be Paid* scale.
(d). Read the value under the hairline on the 3.5% scale: 7 years and 1 month = 85 months



(Fig. 3)

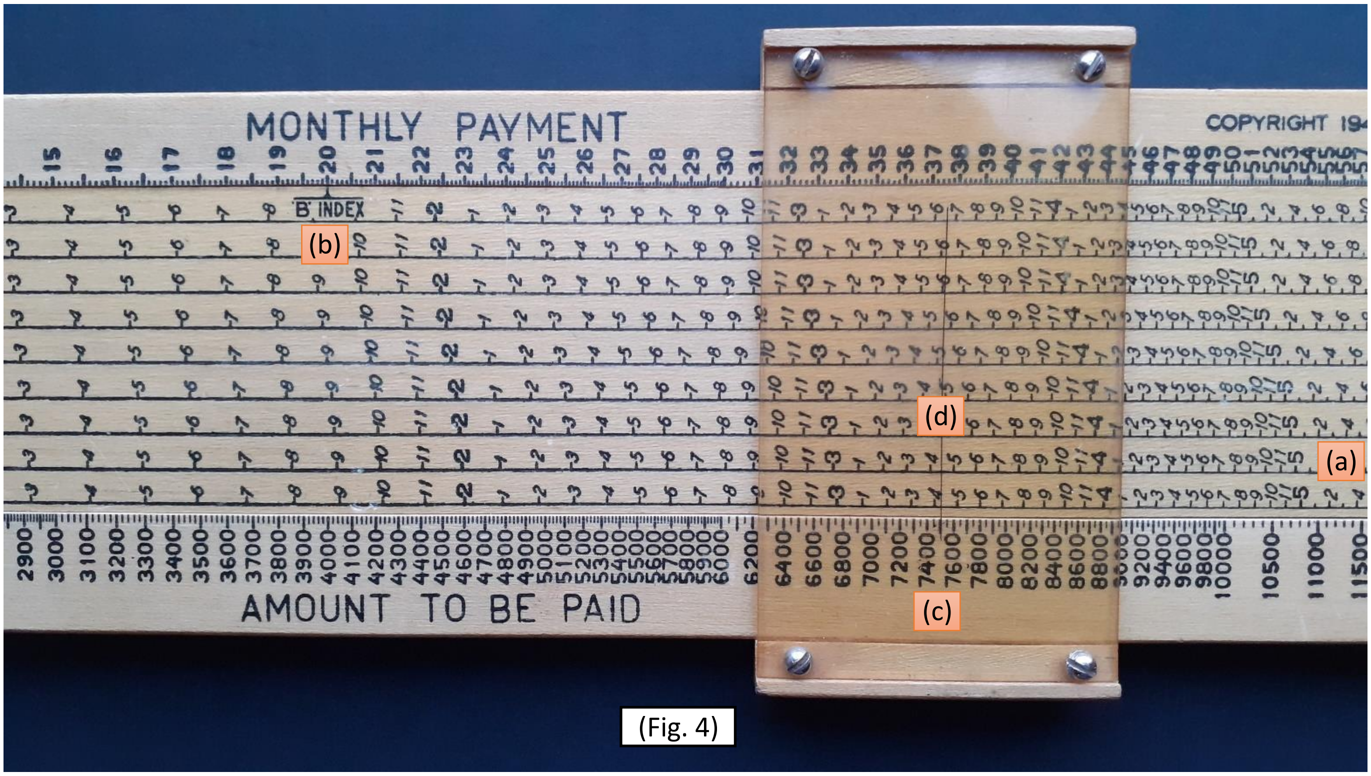
4. How many monthly payments are required to amortize a loan of $750.00, paying $20.00 each month including an annual interest of 4.5%

Using again directly the formula (3) and a calculator:

$$n = -\frac{\ln\left(1 - \dfrac{750 \times 0.045/12}{20}\right)}{\ln(1+.045/12)} = 40.489\,months$$

The result for this problem is obtained with the Lee's rule (Fig. 4) after two simple steps (one setting):

(a). The slider has to display the 4*.5%* scale.
(b). Set the B-index under 20 on the *Monthly Payment* scale
(c). Because $750 is not on the *Amount to be Paid* scale, place the hairline on 7500 (that will be read 750)
(d). Read the value under the hairline on the 4.5% scale: 3 years and 4.5 months = 40.5 months



(Fig. 4)

5. Determine the balance due after 42 payments of $130.00 made for a loan of $16,000 with 3% annual interest

Using formula (3), we first obtain the total monthly payments of $130 necessary to amortize the total loan of $16,000.00 including the 3% annual interest:

$$n = -\frac{\ln\left(1 - \frac{16000 \times 0.03/12}{130}\right)}{\ln(1 + .03/12)} = 147.274\ months$$

Because 42 payments have been done, 105 more payments have to be done to cover the $16,000.00 loan with the 3% annual interest. Using now the formula (1) we can determine the amount that will be paid in the next 105 months:

$$A_n = 130 \times \frac{1 - (1 + 0.03/12)^{-105}}{0.03/12} = \$11{,}992.30$$

The result for this problem is obtained with the Lee's rule (Fig. 5) after five simple steps (two settings):

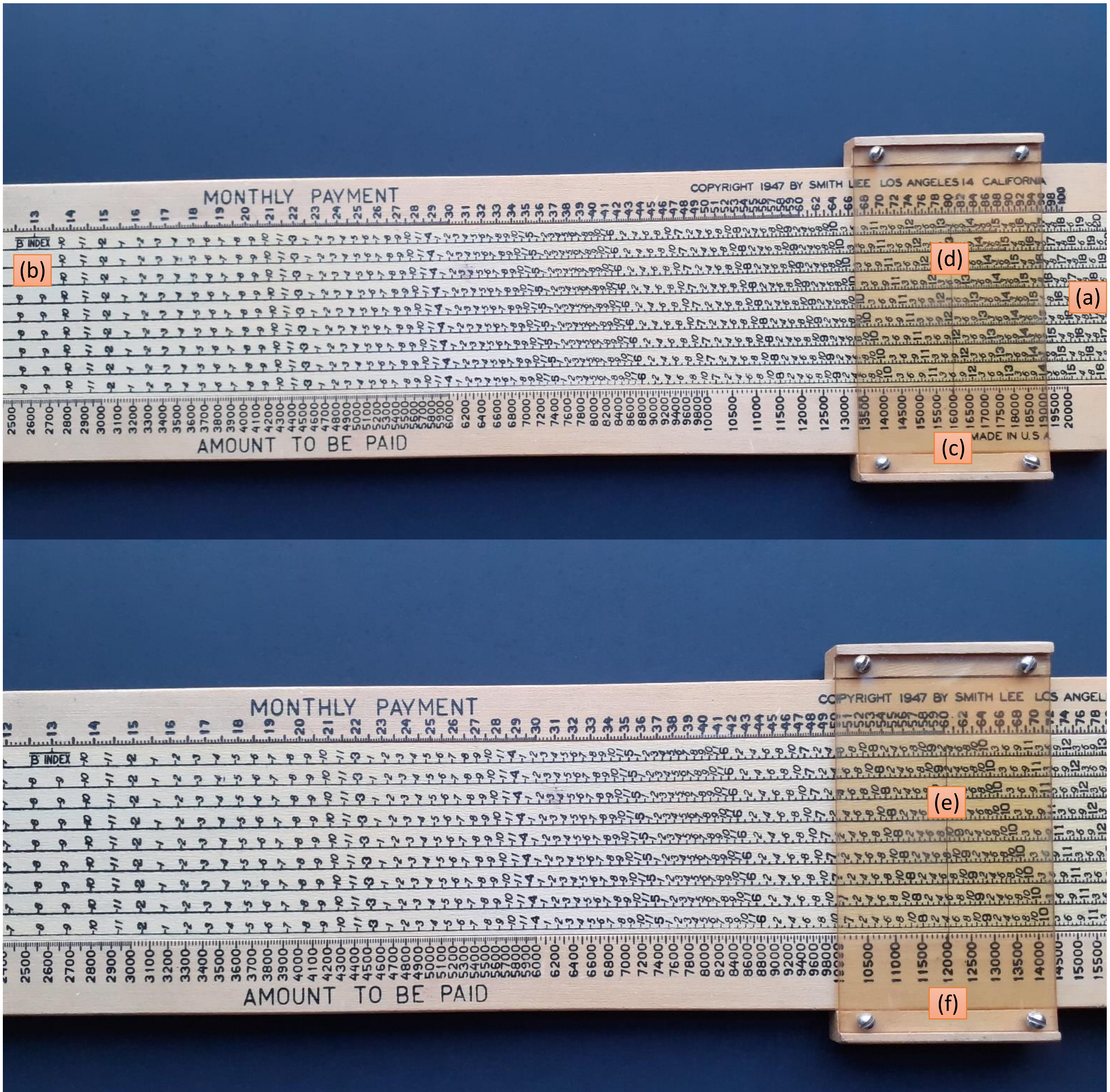
(a). The slider has to display the 3*%* scale.
(b). Because $130 is not on the *Monthly Payment* scale, place the B-index under 13, (that will be read 130)
(c). Place the hairline on $16,000 on the *Amount to Be Paid* scale

(d). Read the value under the hairline on the 3% scale: 12 years and 3 months = 147 months

Subtract the 42 payments done from the total 147 that has to be done: 147 – 42 = 105

(e). Moving <u>only</u> the cursor, place hairline over the 105 (8 years 9 months) on the 3% scale

(f). Read the value under the hairline on the *Amount to Be Paid* scale: $ 12,000



(Fig. 5)

6. How many monthly payments of $1,100.00 are required to reduce a principal of $129,000.00 to $76,000.00, including and annual interest of 4%.

Using formula (3), we first obtain the total monthly payments of $1,100 necessary to amortize the total loan of $129,000.00 including the 4% annual interest:

$$n = -\frac{\ln\left(1 - \dfrac{129000 \times 0.04/12}{1100}\right)}{\ln(1 + .04/12)} = 148.984 \; months$$

Using again formula (3), we obtain the total monthly payments of $1,100 necessary to amortize $76,000.00 of including the 4% annual interest:

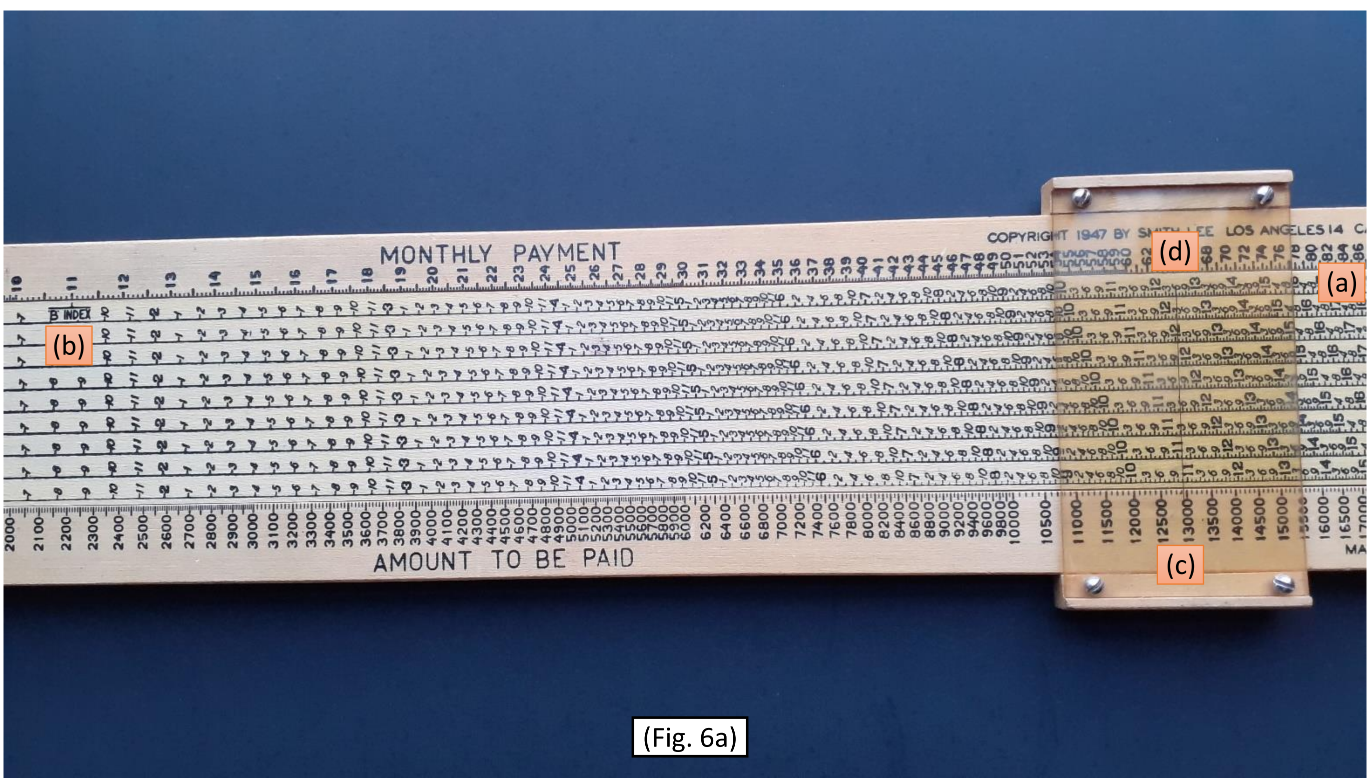
$$n = -\frac{\ln\left(1 - \dfrac{76000 \times 0.04/12}{1100}\right)}{\ln(1 + .04/12)} = 78.658 \; months$$

Subtracting these numbers we find the number of payment necessary to reduce the principal from $129,000 to $76,000.00 including a 4% annual interest:

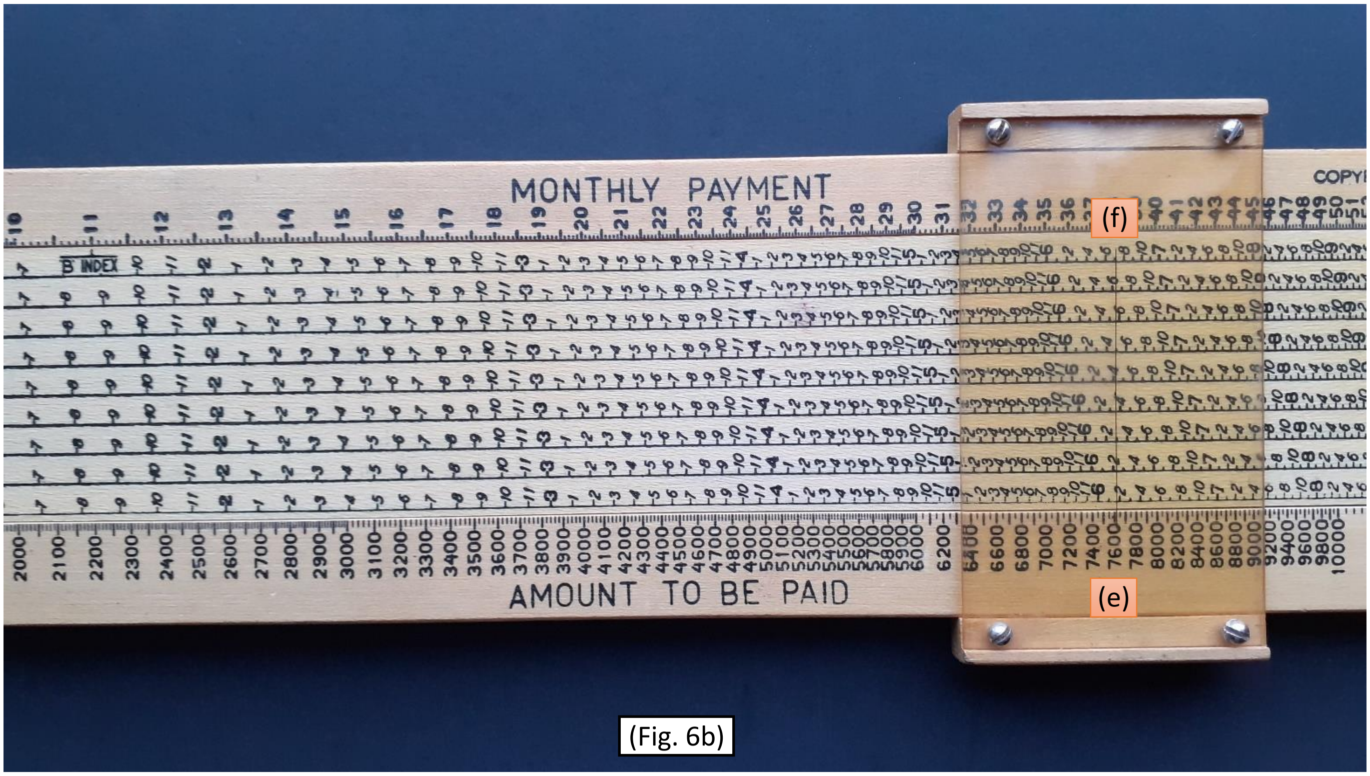
149 – 79 = 70 months

The result for this problem is obtained with the Lee's rule (Fig. 6a, 6b) after four simple steps (two settings):

(a). The slider has to display the *4%* scale.
(b). Because $1100 is not on the *Monthly Payment* scale, place the B-index under 11, (that will be read 1100)
(c). Because $129,000 is not in the *Amount to Be Paid* scale, Place the hairline on $12,900 (that now will be read 129,000)
(d). Read the value under the hairline on the 4% scale: 12 years and 5 months = 149 months
(e). Moving <u>only</u> the cursor, place hairline over the 7,600 on the *Amount to Be Paid* scale (now read 76,000)
(f). Read the value under the hairline on the *4%* scale: 6 years 7 months = 79 months



(Fig. 6a)



(Fig. 6b)

The above examples showed the power of the Lee's Amortizer Rule to solve with a very good precision problems involving complex algebraic and numeric operations that even with a scientific electronic calculator are pretty annoying.

It is also evident that the apparently limitation on the amounts the Lee's rule can handle was inexistent, the logarithmic scales allow to operate the amounts on the scales x10, x100 … same as in a regular slide rule.

Annex A. How to Obtain the Present Value of an Annuity formula

For those very few interested in Math, here is the how formula (1) is obtained. Everything begins with the compound interest formula.

If you invest an amount of money P (principal) at an annual interest rate r, at the end of the first year the accumulated money A_1 is the principal plus the interest paid, the r-portion of the principal:

$$A_1 = P + r \cdot P = P \cdot (1+r)$$

If you invest the accumulated money A_1 again at the same annual interest rate r, at the end of the second year the accumulated money A_2 will be:

$$\begin{aligned} A_2 = A_1 + r \cdot A_1 &= P \cdot (1+r) + r \cdot P \cdot (1+r) \\ &= P \cdot (1+r) \cdot (1+r) \\ &= P \cdot (1+r)^2 \end{aligned}$$

If you invest the accumulated money A_2 again at the same annual interest rate r, at the end of the third year the accumulated money A_3 will be:

$$\begin{aligned} A_3 = A_2 + r \cdot A_2 &= P \cdot (1+r)^2 + r \cdot P \cdot (1+r)^2 \\ &= P \cdot (1+r)^2 \cdot (1+r) \\ &= P \cdot (1+r)^3 \end{aligned}$$

Following the same procedure, if you invest the accumulated money A_3 again n times, at the end of these n-years the accumulated money A_n will be:

$$A_n = P \cdot (1+r)^n \tag{A.1}$$

Formula (A.1) is known as the Compound Interest Formula, and will give you the future value A of a present amount of money P inverted during n years with an annual interest r.

If your investment is done in shorter periods, monthly for instance, formula (A.1) can be adapted to calculate monthly compound interest. For monthly investments, the annual rate r will be divided by 12 months and *n-years* will be converted into months multiplying this number by 12:

$$A_n = P \cdot \left(1 + \frac{r}{12}\right)^{12 \cdot n} \tag{A.2}$$

With formulas (A.1) or (A.2), can be answered several questions. For instance, How much money I have to invest now with an annual interest of 4%, to have at the end of ten years $10,000? In this problem we want to know the Present Value (P) of a Future Amount ($10,000.00), given an interest rate (4%), and a period of time (10 years).

Solving for P in the annual compound formula (A.1), and using the values $A_n = 10{,}000$, $r = 0.04$, and $n = 10$:

$$P = \frac{A_n}{(1+r)^n} = A_n \cdot (1+r)^{-n} \tag{A.3}$$

$$P = \frac{10{,}000}{(1+0.04)^{10}} = \$6{,}755.64$$

If we instead use the Monthly Compound Interest formula (A.2), and the same values:

$$P = \frac{A_n}{\left(1 + \frac{r}{12}\right)^{12n}} = A_n \cdot \left(1 + \frac{r}{12}\right)^{-12n} \tag{A.4}$$

$$P = \frac{10{,}000}{\left(1 + \frac{0.04}{12}\right)^{12 \times 10}} = \$6{,}707.66$$

As we can see, compound interest is more profitable in shorter periods. Formula (A.3) is called in Business and Finance Math *Present Value Formula*. This is the formula we are going to use to obtain the *Present Value of an Annuity* formula (1).

The *Present Value of an Annuity* formula determines the value of a series of future periodic equal payments at a given time. The payments will be deposited at compound interest to yield the specified annuity. The value of an annuity is the sum of the present value of all the payments. Denoting the amount of each payment as m, we can use formula A.3 to find the present value a_1 of the first payment m considering an annual interest r:

$$a_1 = \frac{m}{(1+r)^1} = m \cdot (1+r)^{-n}$$

Using again A.3 the present value a_2 of the second payment can be also calculated:

$$a_2 = \frac{m}{(1+r)^2} = m \cdot (1+r)^{-2}$$

So, formula A.3 can be used to calculate the present value of all the next payments. If n-payments are done, the present value of the annuity is the sum of present values of all the payments done:

$$
\begin{aligned}
A_n &= a_1 + a_2 + \ldots + a_n \\
&= \frac{m}{(1+r)} + \frac{m}{(1+r)^2} + \ldots + \frac{m}{(1+r)^n} \\
&= m \cdot \left[\left(\frac{1}{1+r} \right) + \left(\frac{1}{1+r} \right)^2 + \ldots + \left(\frac{1}{1+r} \right)^n \right]
\end{aligned}
$$

The present value of the annuity is then the sum of powers of a number that is less than one. Using the known formula for Geometric Series:

$$\sum_{i=1}^{n} u^i = \frac{1-u^n}{1-u}, \quad |u| < 1$$

and taking $u = 1/(1 + r)$, the sum giving the present value of the annuity can be simplified doing some basic Algebra:

$$A_n = \frac{1 - \left(\frac{1}{1+r} \right)^n}{1 - \frac{1}{1+r}} = m \times \frac{1-(1-r)^{-n}}{r}$$

obtaining then formula (1)

References.

(1). Rider, Paul R: *College Algebra*, MacMillan Company, New York 1956
(Spanish Version: *Algebra*, Editorial Herrero, Tercera Edicion 1973)

(2). Brechner, Robert A: *Contemporary Mathematics for Business and Consumers*. Second Edition 2000. The Dryden Press, Harcourt Brace College Publishers

(3). www.financeformulas.net: *Finance Formulas – Present Value of Annuity*.
https://financeformulas.net/Present_Value_of_Annuity.html